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THE GALOIS GROUP OF A RECIPROCAL QUARTIC EQUATION.

By DR. L. E. DICKSON, The University of Chicago.

1. The Galois group of a given algebraic equation, $E=0$, gives complete information as to its solvability by radicals, the number and degree of independent root extractions necessary in case the equation is solvable, and reflects the possible factorizations of E within the chosen domain of rationality, R. The Galois group, G, is characterized by the properties that

(*i*) Every rational function of the roots which remains (numerically) unaltered by all the substitutions of G equals a quantity in the domain R;

(*ii*) Every rational function of the roots which equals a quantity in R remains (numerically) unaltered by all the substitutions of G.

Throughout the discussion, the coefficients of every rational function considered are assumed to belong to the domain R.

In determining the group, G, for a given equation, one usually resorts to more or less fortunate guesses as to the rational functions to be employed with success. The present treatment for

$$E\equiv x^4-ax^3+bx^2-ax+1=0 \tag{1}$$

is straightforward and leads in a natural and pleasing way to the criteria desired. While the domain, R, chosen is an arbitrary domain containing a and b, we shall speak of numbers in R as rational, numbers not in R as irrational, and the reader may, if he prefers concreteness, take R to be the domain of all rational (integral or fractional) numbers.

The Galois group for (1) may be any one of ten groups*; necessary and sufficient conditions on a and b for a specified group are determined, and a summary given in § 8.

As the Galois theory is applied only to equations† with distinct roots, we assume that the discriminant Δ of (1) is not zero. By the usual formula, or more simply by § 4, we find that

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*The discussion in the MONTHLY, 1904, p. 195, is incomplete in several respects.

†Multiple factors are first to be determined (by the greatest common divisor process, for example) and removed, as may be done in the domain, R.

$$(2)\qquad \Delta=4AB^2,\ A=(1+\tfrac{1}{2}b)^2-a^2,\ B=a^2-4b+8.$$

Hence we assume throughout the paper that $A\neq 0$, $B\neq 0$.

2. Let the notation for the reciprocal roots be chosen so that

$$(3)\qquad x_1x_2=1,\ x_3x_4=1.$$

If either i or j has a value 1 or 2, the equality $x_ix_j=x_1x_2$ would imply that two x's were equal, contrary to hypothesis. Hence the only products of two roots equal to unity are x_1x_2 and x_3x_4. But x_1x_2 has a rational value. Hence by property (*ii*) of § 1, each substitution of the Galois group, G, either leaves x_1x_2 unaltered formally or else replaces it by x_3x_4. *Hence G is a subgroup of*

$$(4)\qquad G_8 : I,\ (12),\ (34),\ (12)(34),\ (13)(24),\ (14)(23),\ (1324),\ (1423).$$

3. We assume for the present that the equation (1) is irreducible in R, the necessary and sufficient conditions on a, b being given in § 5. Then its group is transitive* and hence either G_8 or one of the following:

$$(5)\qquad G_4 : I,\ (12)(34),\ (13)(24),\ (14)(23);$$
$$(6)\qquad C_4 : I,\ (1324),\ (12)(34).\ (1423).$$

We proceed to determine directly the conditions under which the group is G_4 or C_4. For the latter groups, the identity alone leaves x_1 unaltered, so that (by Lagrange's Theorem) any rational function of the roots equals a rational function of x_1. As x_2 is already expressed as such a function by (3), we seek expressions for x_3 and x_4 as rational functions of x_1. Now

$$(7)\qquad x_3+1/x_3=a-v,\ v\equiv x_1+1/x_1,$$
$$(8)\qquad (x_3-1/x_3)^2=F\equiv(a-v)^2-4.$$

Our aim is therefore to determine the conditions under which F is the square of a rational function of x_1 with coefficients in R:

$$(9)\qquad F=W^2(x_1),\ x_3-1/x_3=W(x_1).$$

To the latter relation we may apply the Galois substitution (12)(34), occurring in both groups (5), (6), and obtain the valid relation

$$x_4-1/x_4=W(x_2),\ \text{or}\ 1/x_3-x_3=W(1/x_1).$$

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*That is, contains substitutions replacing x_1 by each of the x's.

Hence by addition,

$$W(x_1)+W(1/x_1)=0. \tag{10}$$

For the moment, set $V(x_1)=W(x_1)\div(x_1-1/x_1)$. Then $V(1/x_1)=V(x_1)$. Hence $V(x_1)$ equals a rational function of $v=x_1+1/x_1$. But, by (1),

$$v^2-av+b-2=0. \tag{11}$$

The latter is irreducible in R since (1) is irreducible. Hence any rational function of v can be reduced to a linear function. Thus $V(x_1)$ may be given the form $lv+m$, so that

$$W(x_1)=(x_1-1/x_1)(lv+m). \tag{12}$$

The square of the first factor equals v^2-4. Applying (11), we may express W^2 and F, given by (8), as linear functions of v. The latter must be identically equal since (11) is irreducible. Hence

$$l^2(a^3-2ab)+2lm(a^2-b-2)+m^2a=-a, \tag{13}$$
$$l^2(2a^2-a^2b+b^2-4)+2lm(2a-ab)-m^2(b+2)=a^2-b-2. \tag{14}$$

Replacing x_1 by x_3 in (12), we get

$$W(x_3)=(x_3-1/x_3)[l(x_3+1/x_3)+m].$$

Applying (7), (9_2), and (12) to the latter, we get

$$W(x_3)=(x_1-1/x_1)(lv+m)(la+m-lv). \tag{15}$$

First, let the group be C_4. Apply (1324) to (9_2). Thus

$$x_2-1/x_2=W(x_3), \text{ or } 1/x_1-x_1=W(x_3). \tag{16}$$

Inserting the latter value of $W(x_3)$ in (15), we get

$$(lv+m)(la+m-lv)=-1.$$

Eliminating v^2 by means of (11), we get

$$l^2(b-2)+lma+m^2=-1. \tag{17}$$

The determinant of the coefficients of l^2, lm, m^2 in (13), (14), (17) equals $-4AB$ and hence is not zero (§ 1). The unique set of solutions is

$$l^2=\frac{\alpha^2}{4\,AB},\ lm=\frac{-a\alpha\beta}{4\,AB},\ m^2=\frac{a^2\ \beta^2}{4\,AB},$$
$$\alpha\equiv a^2-2b-4,\ \beta\equiv a^2-3b+2.$$

Since l and m are rational, $\surd(AB)$ must be rational. This necessary condition for the group C_4 is also sufficient. Indeed, we get

$$l=\tfrac{1}{2}\,\alpha/\surd(AB),\ m=-\tfrac{1}{2}a\beta/\surd(AB),$$

and all the roots are rational functions of x_1:

$$x_3=\tfrac{1}{2}[a-x_1-\frac{1}{x_1}+W(x_1)],\ x_4=\tfrac{1}{2}[a-x_1-\frac{1}{x_1}-W(x_1)],\tag{18}$$

$$W(x_1)=\frac{1}{2\surd(AB)}(x_1-\frac{1}{x_1})[(a^2-2b-4)(x_1+\frac{1}{x_1})-a^3+3ab-2a].\tag{19}$$

The Galois group, for a domain R, of an irreducible equation (1) *is the cyclic group C_4 if, and only if, $\surd(AB)$ is rational in R.*

Next, let the group be G_4. Apply (13)(24) to (9_2). Then

$$x_1-1/x_1=W(x_3).$$

Since this result differs only in sign from (16_2), we get

$$l^2(b-2)+lma+m^2=+1.\tag{20}$$

The unique set of solutions of (13), (14), (20) is now

$$l^2=\frac{a^2}{4A},\ lm=\frac{a(2+b-a^2)}{4A},\ m^2=\frac{(2+b-a^2)^2}{4A}.$$

Hence $\surd A$ must be rational. Formula (18), with the suitable value of $W(x_1)$ inserted, expresses x_3 as a rational function of x_1.

The Galois group, for a domain R, of an irreducible equation (1) *is the group G_4 if, and only if, $\surd A$ is rational in R.*

4. The last result may be readily verified in various ways. For

$$y_2=x_1x_3+x_2x_4,\ y_3=x_1x_4+x_2x_3,\tag{21}$$

we have, by (3) and the equation (1),

$$y_2+y_3=b-2,\ y_2y_3=x_1^2+x_2^2+x_3^2+x_4^2=a^2-2b.$$

Hence* y_2 and y_3 are the roots of

$$y^2-y(b-2)+a^2-2b=0, \tag{22}$$

$$y_2=\tfrac{1}{2}b-1+\sqrt{A},\quad y_3=\tfrac{1}{2}b-1-\sqrt{A}. \tag{23}$$

Also, if we set $y_1=x_1x_2+x_3x_4=2$,

$$(y_2-y_3)^2=4A,\quad (y_2-2)(y_3-2)=B,\quad \Pi(x_i-x_j)^2=\Pi(y_i-y_j)^2=4AB^2.$$

In particular the three y's are distinct numerically. If $\sqrt{A}$ is irrational, the group cannot be G_4, which leaves y_2 and y_3 unaltered. If $\sqrt{A}$ is rational, the group must be contained in G_4, and being transitive is G_4.

Again, by considering the alternating function $\Pi(x_i-x_j)$, we may conclude that the group is contained in G_4 if, and only if, $\sqrt{\Delta}$ and hence also $\sqrt{A}$ is rational.

5. Consider the question of the reducibility of equation (1) in a domain R. A rational root r, other than ±1, would imply a second root $1/r$ and hence a quadratic factor. This is also the case for $r=\pm1$ as the following argument shows. Suppose that E could have a linear and an irreducible cubic factor:

$$E\equiv(x-r)(x^3+\alpha x^3+\beta x-1/r),\quad r,\ \alpha,\ \beta \text{ rational.}$$

Change x to $1/x$ and multiply the new identity by x^4; then

$$E=(1-rx)(1+\alpha x+\beta x^2-\frac{1}{r}x^3),$$

$$E=(x-\frac{1}{r})(x^3-r\beta x^2-r\alpha x-r).$$

Comparing this with the first expression for E, we get

$$r^2=1,\quad \beta=-r\alpha.$$

The cubic factor would then vanish for $x=r$ and be reducible.

Hence† if E is reducible in R, we may set

$$E\equiv(x^2+px+r)(x^2+qx+1/r), \tag{24}$$

$$p+q=-a,\quad p/r+rq=-a,\quad r+1/r+pq=b. \tag{25}$$

If $r=1$, p and q are rational if and only if B, given by (2), is the square of a rational number, $(p-q)^2$.

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*Or directly from the resolvent cubic of (1), one root being $y_1=2$.

†From this point on we might use Ferrari's three factorizations of a quartic.

If $r=-1$, then $a=0$ and $\sqrt{(-b-2)}$ must be rational.

Finally, let $r^2\neq 1$. The first two equations (25) give

$$p=\frac{-ar}{r+1},\quad q=\frac{-a}{r+1}.$$

For these values, the third condition (25) becomes

$$y+\frac{a^2}{y+2}=b,\quad y\equiv r+\frac{1}{r}. \tag{26}$$

The former is identical with (22) and has the roots (23). For each y, there are two r's whose rationality depends upon the values of

$$y_2^2-4=C_+,\quad y_3^2-4=C_-, \tag{27}$$

$$C_\pm=\tfrac{1}{2}b^2-2-a^2\pm(b-2)\sqrt{A},\quad C_+.C_-=a^2B. \tag{28}$$

If $a\neq 0$, no one of the four values of r is ± 1 (values not falling under the case being treated). If $a=0$, then $C_-=0$ and we have only to consider the two r's connected with $C_+=b^2-4$.

Theorem. *The necessary and sufficient conditions that* (1) *be irreducible in a domain R are that $\sqrt{B}$ be irrational; that either $\sqrt{A}$ be irrational or, in the contrary case, $\sqrt{C_+}$ and $\sqrt{C_-}$ both irrational if $a\neq 0$, $\sqrt{C_+}$ and $\sqrt{(-b-2)}$ both irrational if $a=0$.*

For the case $a=0$, the conditions reduce to the irrationality of

$$\sqrt{(-b+2)},\quad \sqrt{(-b-2)},\quad \sqrt{(b^2-4)}.$$

6. Having determined the Galois group of an irreducible equation (1), we pass to the reducible case. First, let B be the square of a number t of the domain R. We may set

$$x_1+x_2-x_3-x_4=t. \tag{29}$$

Since $t^2=B\neq 0$ (§ 1), the group is a subgroup of

$$H_4\ :\ I,\ (12),\ (34),\ (12)(34), \tag{30}$$

the remaining substitutions in (4) replacing t by $-t$. Now

$$x_1+x_2=\tfrac{1}{2}(a+t),\quad x_3+x_4=\tfrac{1}{2}(a-t),$$

so that the rationality of the x's depends upon

$$(31)\qquad D_{\pm}=(a\pm\sqrt{B})^2-16,\ D_{+}.D_{-}=64A.$$

If $\sqrt{D_+}$ is rational and $\sqrt{D_-}$ is irrational, then x_1 and x_2 alone are rational, and the group is I, (34). If $\sqrt{D_\pm}$ are both irrational and $\sqrt{A}$ rational the group is the common subgroup of H_4, G_4 (see § 8).

7. Finally, let $\sqrt{B}$ be irrational, $\sqrt{A}$ rational, and just one of the pair $\sqrt{C_\pm}$ be rational ($a\neq0$), one of the pair $\sqrt{C_+}$, $\sqrt{(-b-2)}$ rational ($a=0$). Since $\sqrt{A}$ is rational, the group is a subgroup of G_4. Since $\sqrt{B}$ is irrational the group is not contained in $[I, (12)(34)]$, by (29). By (27),

$$C_{+}=(x_1x_3-x_2x_4)^2,\ C_{-}=(x_1x_4-x_2x_3)^2.$$

Hence, for $a\neq0$, the group is composed of the identity and (14)(23) or (13)(24) according as $\sqrt{C_+}$ or $\sqrt{C_-}$ is irrational.

For $a=0$, $y_2=b$, $y_3=-2$. Set $\alpha=x_1+x_4$. Then

$$-\alpha^2=(x_1+x_4)(x_2+x_3)=2+y_2,\ \alpha=\sqrt{(-b-2)}.$$

If α is irrational, the group is I, (13)(24). If α is rational, then $\alpha\sqrt{B}=2\sqrt{(b^2-4)}$ is irrational ($\alpha=0$ would imply $y_2=y_3$, $\Delta=0$), and hence $x_1x_3-x_2x_4$ is irrational, so that the group is I, (14)(23).

8. The following tabular summary gives the necessary and sufficient conditions on the coefficients a and b of the reciprocal quartic (1) that its Galois group for a domain R will be a specified one of the ten subgroups of G_8. For the abbreviations A, B, C, D, see (2), (28), (31); for the first four groups, see (4), (5), (6), (30).

G_8 : $\sqrt{A}$, $\sqrt{B}$, $\sqrt{(AB)}$ all irrational.
C_4 : $\sqrt{B}$ irrational, $\sqrt{(AB)}$ rational.
G_4 : $\sqrt{B}$ irrational, $\sqrt{A}$ rational, $\sqrt{C_\pm}$ both irrational if $a\neq0$; $\sqrt{C_+}\equiv\sqrt{(b^2-4)}$ and $\sqrt{(-b-2)}$ both irrational if $a=0$.
H_4 : $\sqrt{B}$ rational, $\sqrt{D_\pm}$ and $\sqrt{A}$ all irrational.
$\{I,\ (12)(34)\}$: $\sqrt{B}$ and $\sqrt{A}$ rational, $\sqrt{D_\pm}$ both irrational.
$\{I,\ (12)\}$: $\sqrt{B}$ and $\sqrt{D_-}$ rational, $\sqrt{D_+}$ irrational.
$\{I,\ (34)\}$: $\sqrt{B}$ and $\sqrt{D_+}$ rational, $\sqrt{D_-}$ irrational.
$\{I\}$: $\sqrt{B}$ and $\sqrt{D_\pm}$ all rational.
$\{I,\ (13)(24)\}$: $\begin{cases}\sqrt{B}\text{ and }\sqrt{C_-}\text{ irrational, }\sqrt{A}\text{ rational }(a\neq0);\\ \sqrt{B}\text{ and }\sqrt{(-b-2)}\text{ irrational }(a=0).\end{cases}$
$\{I,\ (14)(23)\}$: $\begin{cases}\sqrt{B}\text{ and }\sqrt{C_+}\text{ irrational, }\sqrt{A}\text{ rational }(a\neq0);\\ \sqrt{B}\text{ irrational, }\sqrt{(-b-2)}\text{ rational }(a=0).\end{cases}$

POSTSCRIPT. After my paper had been printed, I noticed that the discussion on pp. 73-74 could be materially simplified by introducing instead of V the function $U(x_1)=W(x_1).(x_1-1/x_1)$. Then $U=Lv+M$. The change amounts to throwing the first factor in (12) into the denominator. In place of (13) and (14) we get

$$L(La+2M)=0,\ L^2(2-b)+M^2=4A.$$

According as the group is C_4 or G_4, we have

$$L(a-v)+M=\mp(Lv+M).$$

Hence for C_4, $La+2M=0$, $L^2B=16A$, so that L and M are rational if and only if $\sqrt{(AB)}$ is rational. For G_4, we have $L=0$, $M^2=4A$, so that $\sqrt{A}$ must be rational. In either case the expression for $W(x_1)$ has x_1-1/x_1 in the denominator, but is much shorter than that in the paper.

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ON CERTAIN TRANSCENDENTAL FUNCTIONS DEFINED BY A SYMBOLIC EQUATION.*

By PROFESSOR R. D. CARMICHAEL, Anniston, Alabama.

1. Introduction and Definition.

The object of this paper is to define a certain large class of transcendental functions of real variables to which the previous consideration of a geometric subject† gave rise and to point out a few of their general properties. Attention will be confined chiefly to questions of continuity and development in series.

We shall take

(1) $$u=f(x) \text{ and } v=f(y),$$

and put

(2) $$u^v=v^u;$$

thus we have an equation which is satisfied identically when x is put for y; and therefore one solution of the symbolic equation (2) is $y=x$, whatever function is denoted by f. We discard at once this solution as not pertinent to the present discussion.

In order to obtain a second solution, put

(3) $$v=\mu u,$$

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*Read before the American Mathematical Society, September 5, 1907.

†Carmichael, The American Mathematical Monthly, Vol. XIII, pp. 221-226, 1906.